Problem 2

A function f is defined by

$$f(x) = \lim_{n\to\infty} \frac{x^{2n} - 1}{x^{2n} + 1}$$

Where is f continuous?

Solution

Rewrite the function like so.

$$f(x) = \lim_{n\to\infty} \frac{(x^2)^n - 1}{(x^2)^n + 1}$$

The limit of a quotient is the quotient of the limits, provided that the limit of the denominator is not equal to 0.

$$f(x) = \frac{\lim\limits_{n\to\infty} \left[(x^2)^n - 1\right]}{\lim\limits_{n\to\infty} \left[(x^2)^n + 1\right]}$$

The limit of a difference (sum) is the difference (sum) of the limits.

$$f(x) = \frac{\lim\limits_{n\to\infty} (x^2)^n - \lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} (x^2)^n + \lim\limits_{n\to\infty} 1}$$

The limit of a constant is just the constant.

$$f(x) = \frac{\lim\limits_{n\to\infty} (x^2)^n - 1}{\lim\limits_{n\to\infty} (x^2)^n + 1}$$

The limit of $(x^2)^n$ as $n \to \infty$ results in different values, depending on what x^2 is. There are three cases to consider.

$$\begin{aligned}
&\text{When } x^2 < 1, \quad \lim_{n\to\infty} (x^2)^n = 0 \\
&\text{When } x^2 = 1, \quad \lim_{n\to\infty} (x^2)^n = \text{undefined} \\
&\text{When } x^2 > 1, \quad \lim_{n\to\infty} (x^2)^n = \infty
\end{aligned}$$

Thus, the piecewise representation of $f(x)$ is

$$f(x) = \begin{cases} -1 & x^2 < 1 \\ \text{undefined} & x^2 = 1 \\ 1 & x^2 > 1 \end{cases}.$$

The points of discontinuity occur where $x^2 = 1$, that is, when $x = \pm 1$. Therefore, f is continuous for the following values of x.

$$\{x \mid x \neq \pm 1\}$$

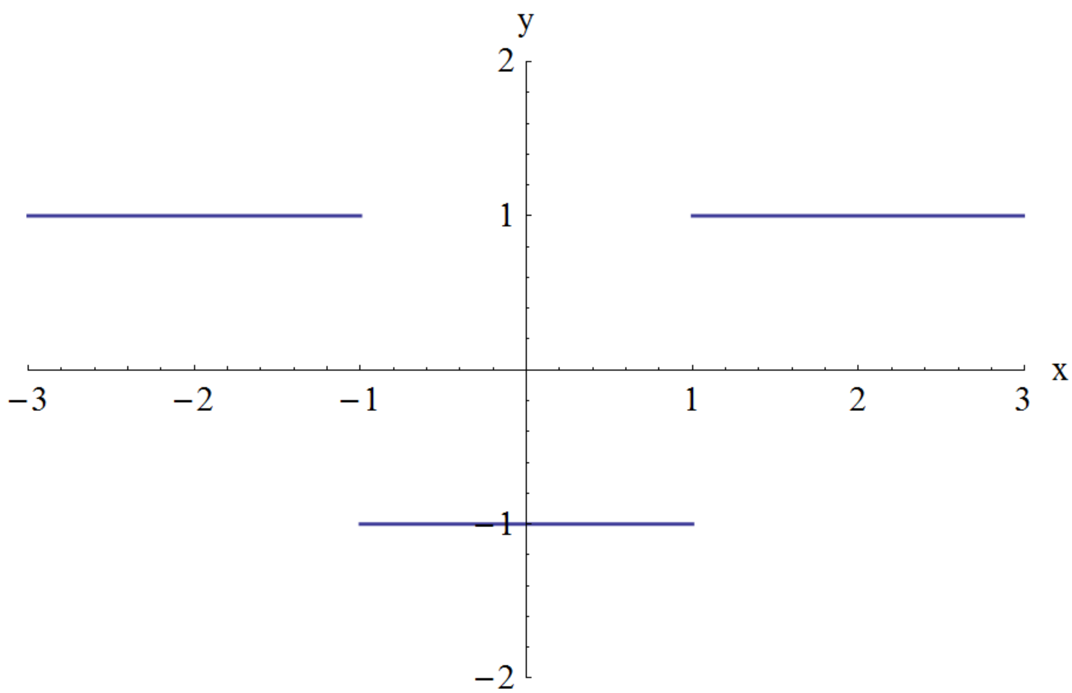

Figure 1: This is what $f(x)$ looks like for $-3 < x < 3$.